

# The Maxwell and Navier-Stokes Equations that Follow from Einstein Equation in a Spacetime Containing a Killing Vector Field

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In this paper we are concerned to reveal that any spacetime structure  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$ , which is a model of a gravitational field in General Relativity generated by an energy-momentum tensor  $\mathbf{T}$  — and which contains at least one nontrivial Killing vector field  $\mathbf{A}$  — is such that the 2-form field  $F = dA$  (where  $A = \mathbf{g}(\mathbf{A}, \cdot)$ ) satisfies a Maxwell like equation — with a well determined current that contains a term of the superconducting type— which follows directly from Einstein equation. Moreover, we show that the resulting Maxwell like equations, under an additional condition imposed to the Killing vector field, may be written as a Navier-Stokes like equation as well. As a result, we have a set consisting of Einstein, Maxwell and Navier-Stokes equations that follows sequentially from the first one under precise mathematical conditions and once some identifications about field variables are evinced, as detailed explained throughout the text. We compare and emulate our results with others on the same subject appearing in the literature.

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## I. INTRODUCTION

In General Relativity, a Lorentzian spacetime structure (LSTS)  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  represents a given gravitational field [22], generated by an energy-momentum distribution  $\mathbf{T} \in \sec T_0^2 M$ , which dynamics is determined by Einstein equation. In this paper we assume ab initio that the LSTS  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  is only an effective description of a gravitational field [6, 20], and that in reality all physical fields are to be interpreted in the sense of Faraday and living in a Minkowski spacetime structure  $\langle M = \mathbb{R}^4, \dot{\mathbf{g}}, \dot{D}, \tau_{\dot{\mathbf{g}}}, \uparrow \rangle$ . Under this assumption, our main aim in this paper is to show (Section 2) that when the effective LSTS  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  possess at least one *nontrivial* Killing vector field  $\mathbf{A} \in \sec TM$ , if we denote by  $A = \mathbf{g}(\mathbf{A}, \cdot) \in \sec \wedge^1 T^*M$  the Killing 1-form field, then given the validity of Einstein equation, the field  $F = dA$  satisfies Maxwell like equations with a well determined current<sup>1</sup>. Moreover the Maxwell like equations satisfied by  $F$  are compatible with Einstein equation in a sense explained in the proof of Proposition 1. We moreover delve into this approach and prove (Section 3) that the Maxwell like equations (the one that follows from Einstein equation) may be written as a Navier-Stokes equation for an inviscid fluid, once we identify a relation between the Killing 1-forms  $A$  and<sup>2</sup>  $\dot{A} = \dot{\mathbf{g}}(\mathbf{A}, \cdot)$  and identify the components of  $\dot{F} = d\dot{A}$  to some variables which appear in the Navier-Stokes equation including as *postulate* that the electric like components of  $\dot{F}$  are equal to the components of the Lamb vector field  $\mathbf{l}$  of the Navier-Stokes fluid plus a well determined vector field  $\mathbf{d}$  that we impose to be a gradient of a smooth function  $\chi$ . Of course, not any Killing vector field  $\mathbf{A}$  of the LSTS  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  satisfy that postulate, but as an example we presented in section III shows the postulate has some non trivial realizations. The equation  $d\dot{F} = 0$  then becomes then the Helmholtz equation for conservation of vorticity and since  $F = \dot{F} + G$

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<sup>1</sup> Our proof of this statement done with the Clifford bundle formalism [18] is very simple when compared with standard tensorial methods.

<sup>2</sup> The 1-form fields  $A$  and  $\dot{A}$  are related by  $A = g(\dot{A})$  and  $F = dg(\dot{A})$  where  $g$  is a well defined extensor field [6, 18] (see below).

(Remark 7) where  $G$  is a closed 2-form field, the homogeneous Maxwell like equation for  $F$  is also equivalent to the Helmholtz equation for conservation of vorticity. In addition, taking again into account that the Navier-Stokes fluid flows in Minkowski spacetime structure and that the LSTS structure  $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$  is an effective one according to ideas developed in [6, 20] the non homogeneous Maxwell like equation for  $F$  (and thus also the one for  $\dot{F}$ ) written in terms of the flat connection  $\tilde{D}$  results in a set of algebraic equations (Eq.(41)) for the components of  $A$  (or for  $\dot{A}$ ), which constrain the identification of its components to the fields appearing in the Navier-Stokes and Einstein equations since the energy-momentum tensor of the matter field gets related to the field variables associated to the Navier-Stokes equation model.

To summarize, we find Maxwell (like) and Navier-Stokes (like) equations that encodes in precise mathematical sense discussed below the contents of Einstein equation when some well defined conditions are satisfied for a given arbitrary nontrivial Killing vector field of the effective LSTS  $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$ . In Section 4 we present our conclusions, where our achievements are briefly compared with other proposals to identify a correspondence between solutions of Einstein and Navier-Stokes equations in spacetimes of different dimensions.

## II. THE MAXWELL LIKE EQUATION EQUIVALENT TO EINSTEIN EQUATION

In this Section we prove two lemmata, a proposition and a corollary whose objective is to obtain given the conditions satisfied by  $A$  mentioned in the introduction a Maxwell like equations for the electromagnetic *like* field  $F = dA$  (where  $A = \mathbf{g}(\mathbf{A}, \cdot)$ ) with a well determined current like term. More precisely we have

**Proposition 1** *Let  $A = \mathbf{g}(\mathbf{A}, \cdot) \in \sec \bigwedge^1 T^*M$  where  $\mathbf{A}$  is a nontrivial Killing vector field in a LSTS  $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$  which represents a gravitational field generated by a given energy-momentum distribution  $\mathbf{T} \in \sec T_0^2 M$  according to Einstein equation. Then  $A$  satisfies the wave equation*

$$\square A - \frac{R}{2}A = -\mathbf{T}(A) \quad (1)$$

where  $\square$  is the covariant D'Alembertian<sup>3</sup>,  $R$  is the scalar curvature,  $\mathbf{T}(A) := \mathbf{T}^\mu A_\mu \in \sec \bigwedge^1 T^*M$ , where the  $\mathbf{T}^\mu = T_\nu^\mu \vartheta^\nu \in \sec \bigwedge^1 T^*M$  are the energy-momentum 1-form fields, with  $\mathbf{T} = T_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu$ . Moreover Eq.(1) is always compatible with Einstein equation<sup>4</sup>.

Moreover, denoting  $F = dA$  and by  $\delta_g$  the Hodge coderivative operator, we have the

**Corollary 2**

$$dF = 0, \quad \delta_g F = -RA + 2\mathbf{T}(A).$$

Before proceeding, note that the field  $F \in \sec \bigwedge^2 T^*M$  satisfies Maxwell equations with a current  $J = RA - 2\mathbf{T}(A)$  that splits in a part  $J_s = RA$ , of the “superconducting” type.

In order to present proofs for the propositions above, the bundle of differential forms is embedded in the Clifford bundle<sup>5</sup> —  $\bigwedge T^*M = \bigoplus_{r=0}^4 \bigwedge^r T^*M \hookrightarrow \mathcal{Cl}(M, g)$ , where  $\mathcal{Cl}(M, g)$  is the Clifford bundle of differential forms [18] where  $g$  is the metric of the cotangent bundle. In this way, given any basis  $\{e_\mu\}$  for  $TU$  ( $U \subset M$ ) with dual basis  $\{\vartheta^\nu\}$  (for  $\bigwedge^1 T^*M = T^*M$ ) then  $\mathbf{g} = g_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu = g^{\mu\nu} \vartheta_\mu \otimes \vartheta_\nu$ ,  $g = g^{\mu\nu} e_\mu \otimes e_\nu = g_{\mu\nu} e^\mu \otimes e^\nu$  and  $g^{\mu\alpha} g_{\alpha\nu} = \delta_\nu^\mu$ . The  $\{\vartheta_\mu\}$  [ $\{e^\mu\}$ ] is the reciprocal basis of  $\{\vartheta^\nu\}$  [ $\{e_\nu\}$ ], namely  $g(\vartheta^\mu, \vartheta_\nu) = \delta_\nu^\mu$  [ $g(e^\mu, e_\nu) = \delta_\nu^\mu$ ] [13].

To start the proof of the propositions we need two lemmata.

**Lemma 3** *If a vector field  $\mathbf{A} \in \sec TM$ , with  $M$  part of the structure  $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$  is a Killing vector field then  $\delta_g A = 0$ , where  $A = \mathbf{g}(\mathbf{A}, \cdot) = A_\mu \vartheta^\mu = A^\mu \vartheta_\mu$  [19].*

<sup>3</sup> In this paper we use the nomenclature and (whenever possible) the notations in [18]. The covariant D'Alembertian is  $\square = \boldsymbol{\partial} \cdot \boldsymbol{\partial}$  where  $\boldsymbol{\partial} = \vartheta^\mu D_{e_\mu}$  is the Dirac operator acting on sections of the Clifford bundle  $\mathcal{Cl}(M, g)$ .

<sup>4</sup> Keep in mind that the validity of Einstein equation is an hypothesis in the proposition.

<sup>5</sup>  $\mathcal{A} \hookrightarrow \mathcal{B}$  means that  $\mathcal{A}$  is embedded in  $\mathcal{B}$  and  $\mathcal{A} \subseteq \mathcal{B}$ .

**Proof.** To prove the Lemma 3 it is only necessary to recall that

$$\mathcal{L}_{\mathbf{A}}g = 0 \iff D_\mu A_\nu + D_\nu A_\mu = 0 \quad (2)$$

and that in the Clifford bundle formalism [18] it follows that  $\delta A = -\partial \lrcorner A$ . Then it reads

$$\begin{aligned} \delta A &= -\vartheta^\mu \lrcorner D_{e_\mu} A = -\vartheta^\mu \lrcorner [(D_\mu A_\nu) \vartheta^\nu] \\ &= g^{\mu\nu} D_\mu A_\nu = \frac{1}{2} g^{\mu\nu} (D_\mu A_\nu + D_\nu A_\mu) = 0, \end{aligned}$$

and the lemma is proved. ■

**Lemma 4** *If  $\mathbf{A} \in \sec TM$  (where  $M$  is part of the structure  $\langle M, g, D, \tau_g, \uparrow \rangle$ ) is a Killing vector field then we have*

$$\partial \wedge \partial A = \square A = \mathcal{R}^\mu A_\mu, \quad (3)$$

where  $\partial = \vartheta^\mu D_{e_\mu}$  is the Dirac operator acting on the sections of the Clifford bundle  $\mathcal{C}\ell(M, g)$  and  $\partial \wedge \partial$  is the Ricci operator acting on  $\sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, g)$ . Finally  $\mathcal{R}^\mu \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, g)$  are the Ricci 1-form fields, with  $\mathcal{R}^\mu = R^\mu_\nu \vartheta^\nu$ , where  $R^\mu_\nu$  are the components of the Ricci tensor.

**Proof.** To prove that  $\partial \wedge \partial A = \mathcal{R}^\mu A_\mu$ , it is well known that the Ricci operator is an *extensorial* entity [18], namely it satisfies  $\partial \wedge \partial A = A_\mu \partial \wedge \partial \vartheta^\mu$ , and since<sup>6</sup>  $\partial \wedge \partial \vartheta^\mu = \mathcal{R}^\mu$  it reads

$$\partial \wedge \partial A = \mathcal{R}^\mu A_\mu.$$

In order to prove that  $\square A = \mathcal{R}^\mu A_\mu$  the definition of the covariant D'Alembertian is used [18], and it follows that

$$\partial \cdot \partial A = g^{\sigma\nu} D_\sigma D_\nu A_\mu \vartheta^\mu$$

Now, the term  $D_\sigma D_\nu A_\alpha$  is calculated. Since  $\mathbf{A}$  is a Killing vector field satisfying Eq.(2) it is possible to write

$$\begin{aligned} D_\sigma (D_\nu A_\mu + D_\mu A_\nu) &= [D_\sigma, D_\nu] A_\mu + D_\nu D_\sigma A_\mu + [D_\sigma, D_\mu] A_\nu + D_\mu D_\sigma A_\nu \\ &= 0. \end{aligned} \quad (4)$$

Taking into account that

$$\begin{aligned} g^{\sigma\nu} [D_\sigma, D_\nu] A_\mu &= 0, \\ g^{\sigma\nu} D_\mu D_\sigma A_\nu &= \frac{1}{2} g^{\sigma\nu} D_\mu (D_\sigma A_\nu + D_\nu A_\sigma) = 0, \end{aligned}$$

and in addition that

$$\begin{aligned} g^{\sigma\nu} [D_\sigma, D_\mu] A_\nu &= -g^{\sigma\nu} R_{\nu}{}^\rho{}_{\sigma\mu} A_\rho = -g^{\sigma\nu} R_{\nu\rho\sigma\mu} A^\rho \\ &= -g^{\sigma\nu} R_{\rho\nu\mu\sigma} A^\rho = -R_{\rho\mu} A^\rho, \end{aligned} \quad (5)$$

multiplying Eq.(4) by  $g^{\sigma\nu}$  it follows that

$$g^{\sigma\nu} D_\nu D_\sigma A_\mu = R_{\rho\mu} A^\rho,$$

and thus

$$\partial \cdot \partial A = g^{\sigma\nu} D_\sigma D_\nu A_\mu \vartheta^\mu = R_{\rho\mu} A^\rho \vartheta^\mu = A^\rho \mathcal{R}_\rho,$$

which proves the Lemma. ■

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<sup>6</sup> See Chapter 4 of [18].

Now all pre-requisites necessary to prove Proposition 1 are demonstrated and then its proof is provided in what follows:

**Proof.** (of Proposition1) Under the hypothesis of Proposition 1, Einstein equation (in geometrical units) is written as  $Ricci - \frac{1}{2}Rg = -\mathbf{T}$  ( $Ricci = R_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu$ ) and can be rewritten in the equivalent form

$$\mathcal{R}^\mu - \frac{1}{2}R\vartheta^\mu = -\mathbf{T}^\mu. \quad (6)$$

Now, we use the fact that the Ricci operator satisfies  $\partial \wedge \partial \vartheta^\mu = \mathcal{R}^\mu$  and moreover that it is an extensorial operator<sup>7</sup>, i. e.,  $A_\mu(\partial \wedge \partial \vartheta^\mu) = \partial \wedge \partial A$ . After multiplying Eq.(6) by  $A_\mu$  it follows that

$$\partial \wedge \partial A - \frac{1}{2}RA = -\mathbf{T}(A), \quad (7)$$

where  $\mathbf{T}(A) = \mathbf{T}^\mu A_\mu \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, g)$ . Now using Eq.(3) which state that  $\partial \wedge \partial A = \square A$  it reads

$$\square A - \frac{1}{2}RA = -\mathbf{T}(A),$$

which proves the first part of the proposition. To prove that Eq.(1) is compatible with Einstein equation we start using Eq.(3), i.e.,  $\partial \wedge \partial A = \square A = \mathcal{R}^\mu A_\mu$ . Eq.(1) can be written after some trivial algebra as  $(R'_\nu - \frac{1}{2}R\delta'_\nu + T'_\nu)A_\mu = 0$ . Next we observe that even if some of the  $A_\mu$  are zero in a given coordinate basis it is always possible to find a new coordinate basis where all the  $A'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A_\mu \neq 0$ . In this new basis we have

$$(R'_\nu - \frac{1}{2}R\delta'_\nu + T'_\nu)A'_\mu = 0. \quad (8)$$

Now, Eq. (8) is a homogeneous system of linear equations for the variables  $A'_\mu$  and since all the  $A'_\mu \neq 0$ , it is necessary that  $\det(R'_\nu - \frac{1}{2}R\delta'_\nu + T'_\nu) = 0$ . It cannot be the case that  $R'_\nu - \frac{1}{2}R\delta'_\nu + T'_\nu \neq 0$ , for the validity of Einstein equation is assumed as hypothesis. So, it follows that Eq.(1) is compatible with the validity of Einstein equation. ■

**Proof.** (of Corollary 2) To prove the corollary we sum  $\square A = \partial \cdot \partial A$  to both members of Eq.(7) and take into account that for any  $\mathcal{C} \in \sec \mathcal{C}\ell(M, g)$  the following expression [18]  $\partial^2 \mathcal{C} = \partial \wedge \partial \mathcal{C} + \partial \cdot \partial \mathcal{C}$  holds. Then

$$\partial^2 A = \frac{1}{2}RA - \mathbf{T}(A) + \square A.$$

Now, since  $\partial^2 A = -\delta \underset{g}{d}A - d\underset{g}{\delta}A$  and Lemma 3 implies that  $\underset{g}{\delta}A = 0$ , it follows that  $\partial^2 A = -\underset{g}{\delta}F$ . Finally, taking into account Eq.(3) it follows that

$$\underset{g}{\delta}F = -\mathbf{J}. \quad (9)$$

Of course,  $\mathbf{J} = -\underset{g}{\delta}F = \partial^2 A = \partial \cdot \partial A + \partial \wedge \partial A = 2\mathcal{R}^\mu A_\mu$  which is equivalent to Eq.(1.7) in Papapetrou paper [16] obtained in component form. See also Eq.(5) of [5]. When Einstein equation is valid we can immediately write taking into account Eq.(6)

$$\mathbf{J} = RA - 2\mathbf{T}(A) \quad (10)$$

and the corollary is proved. ■

**Remark 5 .** We remark that since  $\partial = d - \underset{g}{\delta}$  we can write a single Maxwell like equation<sup>8</sup> for the field  $F$  associated to the Killing form  $A$ , i.e.,

$$\partial F = RA - 2\mathbf{T}(A). \quad (11)$$

<sup>7</sup> Note that the covariant D'Alembertian  $\partial \cdot \partial$  is not an extensorial operator, i.e., in general  $\partial \cdot \partial A \neq A_\mu \partial \cdot \partial \vartheta^\mu$ .

<sup>8</sup> No misprint here!

In [6] it was shown that if the manifold  $M$  is *parallelizable*<sup>9</sup>, i. e., there exists four global vector fields  $\mathbf{e}_a \in \sec TM$ ,  $a = 0, 1, 2, 3$  with  $\{\mathbf{e}_a\}$  a basis for  $TM$ . Take  $\{\mathbf{g}^a\}$  as the dual basis of the  $\{\mathbf{e}_a\}$ . If a LSTS  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  is introduced by postulating that  $\mathbf{g} = \eta_{ab}\mathbf{g}^a \otimes \mathbf{g}^b$ , then the gravitational field is described by field equations — equivalent in a precise mathematical sense to Einstein equation — satisfied by the *potentials*  $\mathbf{g}^a$ . In addition, they are derived through a variational principle from a Lagrangian density

$$\mathcal{L}_g = -\frac{1}{2}d\mathbf{g}^a \wedge \star d\mathbf{g}_a + \frac{1}{2}\delta_g \mathbf{g}^a \wedge \star \delta_g \mathbf{g}_a + \frac{1}{4}d\mathbf{g}^a \wedge \mathbf{g}_a \wedge \star (d\mathbf{g}^a \wedge \mathbf{g}_a). \quad (12)$$

The field equations for the fields  $\mathcal{F}^a = d\mathbf{g}^a \in \sec \bigwedge^2 T^*M$  are:

$$d\mathcal{F}^a = 0, \quad \delta_g \mathcal{F}^a = -(\mathbf{t}^a - \mathbf{T}^a), \quad (13)$$

where the  $\mathbf{T}^a$ , as above, are the energy-momentum 1-form fields of the matter fields and the  $\mathbf{t}^a$  are energy-momentum 1-form fields of the gravitational field. They are indeed *legitimate tensor objects* since in [20] it has been proved that they have the very nice and straightforward expression

$$\mathbf{t}^a = (\partial \cdot \partial)\mathbf{g}^a + \frac{1}{2}R\mathbf{g}^a. \quad (14)$$

Under the conditions above, if Eq.(6) is rewritten in the orthonormal cobasis  $\{\mathbf{g}^a\}$  it reads

$$\mathcal{R}^a - \frac{1}{2}R\mathbf{g}^a = -\mathbf{T}^a.$$

Then, as above, taking into account the definitions of the Ricci, the covariant D'Alembertian, and the Hodge Laplacian operators, together with Eq.(14) and denoting  $\mathbf{t}(A) := \mathbf{t}^a A_a - A_a(\partial \cdot \partial)\mathbf{g}^a$ , the equations of motion of our theory under those conditions are expressed:

$$dF = 0, \quad \delta_g F = -2(\mathbf{t}(A) - \mathbf{T}(A)) \quad (15)$$

that can be summarized in a single equation with the use of the Dirac operator  $\partial$  acting on sections of the Clifford bundle:

$$\partial F = 2(\mathbf{t}(A) - \mathbf{T}(A)). \quad (16)$$

### III. FROM MAXWELL EQUATION TO A NAVIER-STOKES EQUATION

In this Section we obtain a Navier-Stokes equation that follows from the Maxwell like equation obtained above (that as just showed above encodes Einstein equation) once we impose that the electric like components of  $F = dA$  satisfy Eq.(22). In order to do that we start with the observation that the original Navier-Stokes equation describes the non relativistic motion of a general fluid in Newtonian spacetime. It is not thus adequate to use — at least in principle — a general Lorentzian spacetime structure  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  to describe a fluid motion. In fact we want to describe a fluid motion in a background spacetime such that the fluid medium, together with its dynamics, is equivalent to a Lorentzian spacetime governed by Einstein equation in the sense described below.

In order to proceed, it was proposed in [6] a theory of the gravitational field, where gravitation is interpreted as a plastic distortion of the Lorentz vacuum. In that theory the gravitational field is represented by a  $(1,1)$ -extensor field  $\mathbf{h} : \sec \bigwedge^1 T^*M \rightarrow \sec \bigwedge^1 T^*M$  living in Minkowski spacetime<sup>10</sup>. The field  $\mathbf{h}$  — generated by a given energy-momentum distribution in some region  $U$  of Minkowski spacetime — distorts the Lorentz vacuum described by the global cobasis<sup>11</sup>  $\{\gamma^\mu = d\mathbf{x}^\mu\}$ , dual with respect to the basis  $\{\mathbf{e}_\mu = \partial/\partial\mathbf{x}^\mu\}$  of  $TM$ , thus generating the gravitational potentials  $\mathbf{g}^a = \mathbf{h}(\delta_\mu^a \gamma^\mu)$ .

<sup>9</sup> The motivation being Geroch theorem [9] which says that a necessary and sufficient condition for a 4-dimensional Lorentzian manifold  $\langle M, \mathbf{g} \rangle$  to admit spinor fields is that the orthonormal frame bundle be trivial, which implies that the manifold is parallelizable.

<sup>10</sup> Minkowski spacetime is the structure  $\langle M = \mathbb{R}^4, \overset{\circ}{\mathbf{g}}, \overset{\circ}{D}, \tau_{\overset{\circ}{g}}, \uparrow \rangle$ , where  $\overset{\circ}{\mathbf{g}}$  is Minkowski metric,  $\overset{\circ}{D}$  is its Levi-Civita connection, and the remaining symbols define the spacetime orientation and the time orientation.

<sup>11</sup> The  $\langle \mathbf{x}^\mu \rangle$  are global coordinate functions in Einstein-Lorentz Poincaré gauge for the Minkowski spacetime structure that are naturally adapted to an inertial reference frame  $\mathbf{e}_0 = \partial/\partial\mathbf{x}^0, \overset{\circ}{D}\mathbf{e}_0 = 0$ . More details on the concept of reference frames can be found, e.g., in [8, 18]

Now, in the inertial reference frame  $\mathbf{e}_0 = \partial/\partial \mathbf{x}^0$  (according to the Minkowski spacetime structure  $\langle M = \mathbb{R}^4, \dot{\mathbf{g}}, \dot{D}, \tau_{\dot{\mathbf{g}}}, \uparrow \rangle$ ), we write using the global coordinate functions  $\langle \mathbf{x}^\mu \rangle$  for  $M \simeq \mathbb{R}^4$ ,<sup>12</sup>

$$\mathbf{A} = \dot{A}^\mu \mathbf{e}_\mu := \left( \frac{1}{\sqrt{1 - \mathbf{v}^2}} + V_0 + q \right) \mathbf{e}_0 - v^i \mathbf{e}_i = \dot{A}_\mu \mathbf{e}^\mu = \phi \mathbf{e}^0 - v_i \mathbf{e}^i, \quad (17)$$

where the vector function  $\mathbf{v} = (v_1, v_2, v_3)$  is to be identified with the 3-velocity of a Navier-Stokes fluid — in the inertial frame  $\mathbf{e}_0$  according to the conditions disclosed below. Also,  $V_0$  denotes a scalar function representing an external potential acting on the fluid, and

$$q = \int_0^{(t, \mathbf{x})} \frac{dp}{\rho}, \quad (18)$$

where the functions  $p$  and  $\rho$  are identified respectively with the pressure and density of the fluid and supposed functionally related, i.e.,  $dp \wedge dq = 0$ . Furthermore  $\mathbf{v}^2 := \sum_{i=1}^3 (v_i)^2$ .

Before proceeding note that  $\phi$  looks like the relativistic energy per unit mass of the fluid. Then we will write  $\mathbf{A}$  as

$$\mathbf{A} = \left( \frac{1}{2} \mathbf{v}^2 + V + q \right) \mathbf{e}_0 - v^i \mathbf{e}_i = \phi \mathbf{e}^0 - v_i \mathbf{e}^i, \quad (19)$$

where the new potential function  $V$  is the sum of  $V_0$  with the the sum of the Taylor expansion terms of  $[(1 - \mathbf{v}^2)^{-1/2} - \frac{1}{2} \mathbf{v}^2]$ .

Then we have

$$\dot{A} = \dot{\mathbf{g}}(\mathbf{A},) = \dot{A}_\mu \gamma^\mu = \eta_{\mu\nu} \dot{A}^\nu \gamma^\mu = \dot{A}^\mu \gamma_\mu, \quad \dot{F} = d\dot{A} = \frac{1}{2} \dot{F}_{\mu\nu} \gamma^\mu \wedge \gamma^\nu,$$

and

$$A = \mathbf{g}(\mathbf{A},) = A_\mu \gamma^\mu = g_{\mu\nu} \dot{A}^\nu \gamma^\mu = A^\mu \gamma_\mu, \quad F = dA = \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu,$$

We. proceed by identifying the magnetic like and the electric like components of the field  $\dot{F}$  with components of the vorticity field and the Lamb vector field of the fluid. We remark that whereas the identification of the magnetic like components of  $\dot{F}$  with the vorticity field is natural, the identification of the electric like components of  $F$  is here **postulated**, i.e. we suppose that  $\dot{F}$  besides being derived from a Killing vector field associated to the structure also satisfies the constraint given by Eq.(22) below. We thus write  $\dot{F}_{\mu\nu} = (d\dot{A})_{\mu\nu}$  as

$$\dot{F}_{\mu\nu} = \begin{pmatrix} 0 & l_1 - d_1 & l_2 - d_2 & l_3 - d_3 \\ -l_1 + d_1 & 0 & -w_3 & w_2 \\ -l_2 + d_2 & w_3 & 0 & -w_1 \\ -l_3 + d_3 & -w_2 & w_1 & 0 \end{pmatrix} \quad (20)$$

where  $\mathbf{w}$  is vorticity of the velocity field

$$\mathbf{w} := \nabla \times \mathbf{v}, \quad (21)$$

and

$$\mathbf{l} := \mathbf{w} \times \mathbf{v}, \quad (22)$$

is the so called *Lamb* vector and moreover

$$\mathbf{d} = -\nabla \chi, \quad (23)$$

where  $\chi$  is a smooth function.

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<sup>12</sup> The basis  $\{\mathbf{e}^\mu\}$  is the *reciprocal* basis of the basis  $\{\mathbf{e}_\mu\}$ , i.e.,  $\dot{\mathbf{g}}(\mathbf{e}^\mu, \mathbf{e}_\nu) = \delta_\nu^\mu$ .

**Remark 6** Before proceeding it is important to emphasize that the identification of the components of  $\mathring{F}$  with the components of the Lamb and vorticity fields of a fluid selects a preferred reference frame, namely the inertial reference frame  $\mathbf{e}_0 = \partial/\partial \mathbf{x}^0$  introduced above.

At this point we recall that the non relativistic Navier-Stokes equation for an *inviscid fluid* is given by [2, 7]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla(V + q), \quad (24)$$

or using a well known vector identity,

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{w} \times \mathbf{v} - \nabla \left( V + \frac{p}{\rho} + \frac{1}{2} \mathbf{v}^2 \right). \quad (25)$$

By these identifications<sup>13</sup>, we get a Navier-Stokes like equation from the straightforward identification of  $\mathbf{l} - \mathbf{d} = (F_{01}, \mathring{F}_{02}, \mathring{F}_{03})$  and  $\mathbf{w} = (\mathring{F}_{32}, \mathring{F}_{13}, \mathring{F}_{21})$ . Indeed, we have

$$\mathring{F}_{0i} = (\mathbf{w} \times \mathbf{v})_i - d_i = -\frac{\partial v_i}{\partial t} - \frac{\partial \phi}{\partial \mathbf{x}^i}, \quad (26)$$

$$\mathring{F}_{jk} = -\sum_{i=1}^3 \epsilon_{ijk} w_i, \quad (27)$$

where  $\epsilon_{ijk}$  is the 3-dimensional Kronecker symbol. Eq.(26) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \nabla \left( \frac{1}{2} \mathbf{v}^2 \right) = -\nabla \left( V + \frac{p}{\rho} \right) + d_i, \quad (28)$$

and since  $\mathbf{d} = -\nabla \chi$  for some smooth function  $\chi$  then Eq.(28) can be written as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{w} \times \mathbf{v} + \nabla \left( \frac{1}{2} \mathbf{v}^2 \right) = -\nabla \left( V + \chi + \frac{p}{\rho} \right) \quad (29)$$

which is now a Navier-Stokes like equation for a fluid moving in an external potential  $V' = V + \chi$ .

Moreover, the homogeneous Maxwell equation  $d\mathring{F} = 0$  is equivalent to

$$\begin{aligned} \nabla \times \mathbf{l} + \frac{\partial \mathbf{w}}{\partial t} &= 0, \\ \nabla \cdot \mathbf{w} &= 0, \end{aligned} \quad (30)$$

which express Helmholtz equation for conservation of vorticity.

**Remark 7** Note that since  $A = \mathbf{g}(\mathbf{A}, \cdot) = A_\mu \gamma^\mu = \mathring{A}^\mu g_{\mu\nu} \gamma^\nu$ , we can write<sup>14</sup>

$$A = g(\mathring{A}),$$

where the extensor field  $g$  is defined by  $g(\gamma_\mu) = g_{\mu\nu} \gamma^\nu$ . Thus, in general we can write since  $d(F - \mathring{F}) = 0$ ,  $F = \mathring{F} + G$  where  $G$  is a closed 2-form field<sup>15</sup>. So,  $dF = d\mathring{F} = 0$  express the same content, namely Eq.(30), the Helmholtz equation for conservation of vorticity.

**Remark 8** Of course, the main idea of this paper that we can get from the Maxwell like equation that follows from the Einstein equation a Navier-Stokes equation for an inviscid fluid depends on the existence of nontrivial solutions of Eq.(20), i.e.,  $\mathring{F}_{0i} = (\mathbf{w} \times \mathbf{v})_i + d_i$ . So, it is important to show that this equation has nontrivial realizations for at least some Killing vector fields living in  $M$ , such that  $\langle M, \mathbf{g}, D, \tau_g, \uparrow \rangle$  models a gravitational field. That this is

<sup>13</sup> Other identifications of Navier-Stokes equation with Maxwell equations may be found in [24, 25].

<sup>14</sup> We have (details in [6])  $g = \mathbf{h}^\flat \mathbf{h}$  and  $\mathring{g}(g(\gamma_\mu), \gamma_\nu) = g_{\mu\nu} = \mathring{g}(\mathbf{h}(\gamma_\mu), \mathbf{h}(\gamma_\nu)) = \mathring{g}(\mathfrak{g}_\mu, \mathfrak{g}_\nu)$ .

<sup>15</sup> Even more, taking into account that the Minkowski manifold is star shape we have that  $G$  is exact. Thus  $F = \mathring{F} + dK$ , for some smooth 1-form field  $A$ .

the case, is easily seen if we take, e.g., the Schwarzschild spacetime structure. As well known [12] the vector field  $\mathbf{A} = \partial_\varphi = -\mathbf{x}^2 \partial_{\mathbf{x}^1} + \mathbf{x}^1 \partial_{\mathbf{x}^2}$  is a Killing vector field for the Schwarzschild metric<sup>16</sup>. The 1-form field corresponding to it and living in Minkowski spacetime is  $\dot{A} = \mathbf{x}^2 d\mathbf{x}^1 - \mathbf{x}^1 d\mathbf{x}^2$ . Thus  $\phi = 0$  and  $\mathbf{v} = (\mathbf{x}^2, -\mathbf{x}^1, 0)$ . This gives  $0 = \dot{F}_{0i} = -d_i + (\mathbf{w} \times \mathbf{v})_i$ , i.e.,

$$\mathbf{d} = -\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla[(\mathbf{x}^1)^2 + (\mathbf{x}^2)^2], \quad (31)$$

and Eq.(29) holds.

**Remark 9** For the example given in Remark 8 we have simply  $A = f \dot{A}$  where  $f = r^2 \cos^2 \theta$ . Thus in this case, we have the simple expression

$$F = df \wedge \dot{A} + f \dot{F}. \quad (32)$$

There are many examples of Killing vector fields for which  $A = f \dot{A}$  and for such fields that developments given below in terms of  $A$  are easily translated in terms of  $\dot{A}$ . In particular, when  $A = f \dot{A}$  we have the following identification of the components of the Lamb and vorticity vector fields with the components of  $F$ ,

$$\begin{aligned} l_i &= (\mathbf{w} \times \mathbf{v})_i = \frac{1}{f} F_{0i} - (d \ln f \wedge \dot{A})_{0i}, \\ \mathbf{w}_i &= -\frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} \left[ \frac{1}{f} F_{jk} - (d \ln f \wedge \dot{A})_{jk} \right]. \end{aligned} \quad (33)$$

To continue, we recall that in order for the Navier-Stokes equation just obtained to be compatible with Einstein equation it is necessary yet to take into account Eq.(9), the non homogeneous Maxwell equation written in the structure  $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$ , and the fact that  $A$  is in the Lorenz gauge, namely  $\delta_{\mathbf{g}} A = 0$ , since these equations produce as we are going to see constraints among the several fields involved, The constraints involving the components of  $A = \mathbf{g}(\mathbf{A},)$  with  $\mathbf{A}$  defined in Eq.(17) are also encoded in Eq.(1), which *must* now be expressed in terms of the objects defining the Minkowski spacetime structure  $\langle M = \mathbb{R}^4, \dot{\mathbf{g}}, \dot{D}, \tau_{\dot{\mathbf{g}}}, \uparrow \rangle$ .

Now, taking into account that  $\dot{D}\dot{\mathbf{g}} = 0$ , we have

$$D\dot{\mathbf{g}} = \mathcal{A} \in \sec T_0^2 M \otimes \wedge^1 T^* M, \quad (34)$$

where  $\mathcal{A} \in \sec T_0^2 M \otimes \wedge^1 T^* M$  is the non metricity tensor of  $D$  with respect to  $\dot{\mathbf{g}}$ . In the coordinates  $\langle \mathbf{x}^\mu \rangle$  introduced above it follows that

$$\mathcal{A} = Q_{\alpha\beta\sigma} \gamma^\sigma \otimes \gamma^\beta \otimes \gamma^\delta. \quad (35)$$

Then, as it is well known<sup>17</sup> the relation between the coefficients  $\Gamma_{\mu\alpha}^\nu$  and  $\dot{\Gamma}_{\mu\alpha}^\nu$  associated to the connections  $D$  and  $\dot{D}$  ( $D e_\mu \vartheta^\nu = -\Gamma_{\mu\alpha}^\nu \vartheta^\alpha$ ,  $\dot{D} e_\mu \vartheta^\nu = -\dot{\Gamma}_{\mu\alpha}^\nu \vartheta^\alpha$ ) in an arbitrary coordinate vector  $\{\frac{\partial}{\partial x^\mu}\}$  and covector  $\{\vartheta^\nu = dx^\nu\}$  bases — associated to arbitrary coordinate functions  $\{x^\mu\}$  covering  $U \subset M$  — are given by<sup>18</sup>

$$\Gamma_{\mu\alpha}^\nu = \dot{\Gamma}_{\mu\alpha}^\nu + \frac{1}{2} S_{\mu\alpha}^\nu, \quad (36)$$

where

$$S_{\alpha\beta}^\rho = \dot{g}^{\rho\sigma} (Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) \quad (37)$$

are the components of the so called *strain tensor* of the connection.

In the coordinate bases  $\{\frac{\partial}{\partial x^\mu}\}$  and  $\{\gamma^\mu = dx^\mu\}$ , associated to the coordinate functions  $\langle \mathbf{x}^\mu \rangle$ , it follows that  $\dot{\Gamma}_{\mu\alpha}^\nu = 0$  and in addition the following relation for the Ricci tensor of  $D$  holds:

$$R_{\mu\nu} = J_{(\mu\nu)}.$$

<sup>16</sup> The spherical coordinate functions are  $(r, \theta, \varphi)$ .

<sup>17</sup> See, e.g., [18].

<sup>18</sup> We use that  $\dot{\mathbf{g}} = \dot{g}_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu = \dot{g}^{\mu\nu} \vartheta_\mu \otimes \vartheta_\nu$ , where  $\{\vartheta_\mu\}$  is the reciprocal basis of  $\{\vartheta^\mu\}$ , namely  $\vartheta_\mu = \dot{g}_{\alpha\mu} \vartheta^\alpha$  and  $\dot{g}^{\mu\nu} \dot{g}_{\mu\kappa} = \delta_\kappa^\nu$ . In the bases associated to  $\langle \mathbf{x}^\mu \rangle$  it is  $\dot{\mathbf{g}} = \eta_{\mu\nu} \gamma^\mu \otimes \gamma^\nu = \eta^{\mu\nu} \gamma_\mu \otimes \gamma_\nu$  [3, 4].



Denoting  $K_{\alpha\beta}^\rho = -\frac{1}{2}S_{\alpha\beta}^\rho$ , the  $J_{(\mu\nu)}$  is the symmetric part of

$$J_{\mu\alpha} = \dot{D}_\alpha K_{\rho\mu}^\rho - \dot{D}_\rho K_{\alpha\mu}^\rho + K_{\alpha\sigma}^\rho K_{\rho\mu}^\sigma - K_{\rho\sigma}^\rho K_{\alpha\mu}^\sigma.$$

Now, if the Dirac operator associated to the Levi-Civita connection  $\dot{D}$  of  $\dot{\mathbf{g}}$  is introduced by

$$\dot{\phi} := \boldsymbol{\vartheta}^\mu \dot{D}_{\frac{\partial}{\partial x^\mu}} = \gamma^\mu \dot{D}_{\frac{\partial}{\partial x^\mu}} \quad (38)$$

it can be shown that<sup>19</sup>

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = (\dot{\phi} \wedge \dot{\phi}) \dot{A} + \mathbf{L}^\alpha \cdot_{\dot{\mathbf{g}}} \gamma_\alpha \dot{A}, \quad (39)$$

where  $A = A_\mu \gamma^\mu$ ,  $\dot{A}_\kappa := \eta_{\beta\kappa} g^{\beta\sigma} A_\sigma$ , and  $\mathbf{L}^\alpha = \eta^{\alpha\beta} J_{\beta\sigma} \gamma^\sigma$ . The symbol  $\cdot_{\dot{\mathbf{g}}}$  denotes the scalar product accomplished with  $\dot{\mathbf{g}}$ . Since  $\dot{\phi} \wedge \dot{\phi} = \dot{\mathcal{R}}^\sigma \dot{A}_\sigma = \dot{R}_\alpha^\sigma \dot{A}_\sigma \gamma^\alpha = 0$  it reads

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = \eta^{\alpha\beta} J_{\beta\alpha} \dot{A} = \eta^{\alpha\beta} J_{\beta\alpha} \eta_{\nu\kappa} g^{\nu\sigma} A_\sigma \gamma^\kappa. \quad (40)$$

According to Eq.(3)  $\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = \square A$  and thus Eq.(1) can also be written as

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} A = \frac{1}{2} R A - \mathbf{T}(A).$$

Taking into account Eq.(40), the following algebraic equation<sup>20</sup>, relating the components  $A_\sigma$  to the components of the energy-momentum tensor of matter and the components of the  $\mathbf{g}$  field that is part of the original LSTS, is obtained:

$$\eta^{\alpha\beta} J_{\beta\alpha} \eta_{\nu\kappa} g^{\nu\sigma} A_\sigma = \frac{1}{2} g^{\mu\alpha} J_{(\mu\alpha)} A_\kappa - T_\kappa^\sigma A_\sigma. \quad (41)$$

Eq.(41) are the constraints need to be satisfied by the variables of our theory in order for the Navier-Stokes equation to be compatible with the contents of Einstein equation.

As a last remark we observe that Eq.(41) may be also interpreted as an equation providing the energy-momentum tensor of the matter field as a function of the variables entering the Navier-Stokes identification.

#### IV. CONCLUSIONS

We demonstrated that for each Lorentzian spacetime representing a gravitation field in General Relativity which contains an arbitrary Killing vector field  $\mathbf{A}$ , the field  $F = dA$  (where  $A = \mathbf{g}(\mathbf{A}, \cdot)$ ) satisfies Maxwell like equations with well determined current 1-form field. Moreover we showed that for all Killing 1-form fields  $\dot{A} = \dot{\mathbf{g}}(\mathbf{A}, \cdot)$ , when some identifications of the components of  $\dot{A}$  and the variables entering the Navier-Stokes equation are accomplished and in particular when the postulated nontrivial condition (Eq.(20)— $\dot{F}_{0i} = (d\dot{A})_{0i} = l_i - d_i$ — is satisfied, the Maxwell like equations for  $\dot{F}$  and thus the ones for  $F$  can be written as a Navier-Stokes equation representing an *inviscid fluid*. Thus, the Maxwell and Navier-Stokes like equations found in this paper<sup>21</sup> are almost directly obtained from Einstein equation through thoughtful identification of fields. All fields in our approach live in a 4-dimensional background spacetime, namely Minkowski spacetime and the Lorentzian spacetime structures  $\langle M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow \rangle$  is considered only a (sometimes useful) description of gravitational fields. Thus our approach is in contrast with the very interesting and important studies in, e. g., [1, 10, 17] where it is shown through some identifications that every solution for an incompressible Navier-Stokes equation in a  $(p+1)$ -dimensional spacetime gives rise to a solution of Einstein equation in  $(p+2)$ -dimensional spacetime. It is worth also to quote [15] where it is also suggested an interesting relation between Einstein equations and the Navier-Stokes equation. Finally we remark that it is clear that we can find examples [21] of Lorentzian spacetimes that do not have any nontrivial Killing vector field. However, as asserted in Weinberg [26], all Lorentzian spacetimes that represent gravitational fields of physical interest possess some Killing vector fields, and we exhibit an example (Remark 8) the approach of the present paper applies.

<sup>19</sup> See Exercise 291 in [18].

<sup>20</sup> Of course, it is a partial differential equation that needs to be satisfied by the components of the stress tensor of the connection.

<sup>21</sup> In [23] a fluid satisfying a particular Navier-Stokes equation is also shown to be approximately equivalent to Einstein equation. Our approach is completely different from the one in [23].

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